COMBINATORICA
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA 11 (4) (1991) 299–314

EDGE-ISOPERIMETRIC INEQUALITIES IN THE GRID

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Received January 4, 1989

The grid graph is the graph on $[k]^n = \{0, \dots, k-1\}^n$ in which $x = (x_i)_1^n$ is joined to $y = (y_i)_1^n$ if for some i we have $|x_i - y_i| = 1$ and $x_j = y_j$ for all $j \neq i$. In this paper we give a lower bound for the number of edges between a subset of $[k]^n$ of given cardinality and its complement. The bound we obtain is essentially best possible. In particular, we show that if $A \subset [k]^n$ satisfies $k^n/4 \leq |A| \leq 3k^n/4$ then there are at least k^{n-1} edges between A and its complement.

Our result is apparently the first example of an isoperimetric inequality for which the extremal sets do not form a nested family.

We also give a best possible upper bound for the number of edges spanned by a subset of $[k]^n$ of given cardinality. In particular, for $r = 1, \dots, k$ we show that if $A \subset [k]^n$ satisfies $|A| \leq r^n$ then the subgraph of $[k]^n$ induced by A has average degree at most $2n(1 - 1/r)$.

0. Introduction

In this paper we are interested in edge-isoperimetric inequalities on graphs. Given a graph G and a natural number m, at most how many edges are spanned by a set of m vertices? Also, at least how many edges are there between a set of m vertices and its complement?

To make these questions precise, let us introduce a small amount of notation. Given a graph G, define the *edge-boundary* $\partial_e(A)$ and the *edge-interior* $\mathrm{int}_e(A)$ of a set $A \subset V(G)$ by

$$\partial_e(A) = \{xy \in E(G) : \; x \in A, \; y \notin A\},$$
$$\mathrm{Int}_e\,(A) = \{xy \in E(G) : \; x, y \in A\}.$$

Then we wish to determine $\min\{|\partial_e(A)| : \; |A| = m\}$ and $\max\{|\mathrm{Int}_e\,(A)| : \; |A| = m\}$.

If G is a regular graph then these two problems happen to be equivalent. Indeed, if G is r-regular then $|\partial_e(A)| + 2|\mathrm{Int}_e\,(A)| = r|A|$ for any $A \subset G$. For the discrete cube $\mathcal{Q}^n = \mathcal{P}(\{1, \dots, n\})$ (with S joined to T if $|S \Delta T| = 1$), the edge-isoperimetric inequality was determined by Harper [6], Lindsey [9], Bernstein [1] and Hart [7]. In order to state their result, define an order on $\mathcal{Q}^n$, the *binary order* , by letting a set S precede a set T if $\max(S \Delta T) \in T$, in other words if the greatest element of $\{1, \dots, n\}$ which is in one of S and T but not the other is actually in T. With this terminology, Harper, Lindsey, Bernstein and Hart proved that initial segments of the binary order are subsets of $\mathcal{Q}^n$ of minimum edge-boundary (and so maximum edge-interior). See [2, Ch.16] for a general discussion of this and related topics.

Our aim in this paper is to give an answer to these questions for the grid $[k]^n = \{0, \dots, k-1\}^n$, where as usual $x = (x_1, \dots, x_n)$ is joined to $y = (y_1, \dots, y_n)$

AMS subject classification (1991): 05 C 35

* Research partially supported by NSF Grant DMS-8806097

if for some i we have $|x_i - y_i| = 1$ and $x_j = y_j$ for all $j \neq i$. Since $[k]^n$ is not regular (for $k \geq 3$), the two problems are distinct. We give an inequality for each problem: our lower bound on the edge-boundary is best possible for a reasonably large set of values of m for every k and n, while our upper bound on the edge-interior is best possible for all values of m.

We remark that there is a superficial resemblance between the grid we study and the graph on $[k]^n$ in which two points are joined if they differ in precisely one coordinate. For this graph, the edge-isoperimetric problem was solved by Lindsey [9] (see also Clements [4] and Kleitman, Krieger and Rothschild [8]). In spite of the apparent similarity, the isoperimetric problems on these two graphs seem to have very little in common.

Our approach to the problem of minimising the edge-boundary is first to prove a related best possible inequality for subsets of the continuous cube $I^n = [0,1]^n$, and then to use this to deduce a discrete inequality. The continuous problem is actually very natural in its own right. One important point to note is that, for each of these inequalities, the extremal sets do *not* form a nested family: in fact, these are essentially the first examples of isoperimetric inequalities in which the extremal sets are not nested. In other words, there is no ordering on $[k]^n$ with the property that its initial segments, or even a fairly dense family of its initial segments, are extremal. Because of this, a little thought shows that compression operators alone, as often used in proving isoperimetric inequalities (see e.g. [2], [3], [5], [10]), cannot be enough to prove the inequalities.

For the problem of maximising the edge-interior, there is again a continuous analogue which is interesting in its own right. We give a best possible inequality for the continuous problem. From this inequality it is possible to obtain a fairly good inequality in the discrete case. However, we give in addition a best possible inequality for the discrete problem, based on a direct discrete argument. Somewhat surprisingly, it turns out that, although the extremal sets for the discrete problem are nested, those for the continuous analogue are not.

We also consider the discrete torus, that is, the graph on $\mathbb{Z}_k^n = (\mathbb{Z}/k\mathbb{Z})^n$ in which $x = (x_i)_1^n$ is joined to $y = (y_i)_1^n$ if for some i we have $x_i = y_i \pm 1$ and $x_j = y_j$ for all $j \neq i$. As this graph is regular, the two edge-isoperimetric problems coincide. It turns out that we have to do almost no additional work to obtain an essentially best possible inequality for $\mathbb{Z}_k^n$.

Our notation is fairly standard. We regard $[k]^n$ as a subset of $\mathbb{Z}^n$, and we write $e_1, \ldots, e_n$ for the standard basis of $\mathbb{Z}^n$. Thus for example $3e_1 + 2e_3$ denotes the point $(3, 0, 2, 0, \ldots, 0)$. A set $A \subset [k]^n$ is a *down-set* if whenever $x, y \in [k]^n$ with $y \in A$ and $x_i \leq y_i$ for all i then also $x \in A$.

We write I for the unit interval $[0,1] \subset \mathbb{R}$. The Lebesgue measure of a measurable set $A \subset I^n$ is written $m(A)$. Again, a set $A \subset I^n$ is a *down-set* if whenever $x, y \in I^n$ with $y \in A$ and $x_i \leq y_i$ for all i then also $x \in A$.

For $S \subset \{1, \ldots, n\}$, we write $[k]^S$ for $\{x \in [k]^n : x_i = 0 \text{ for } i \notin S\}$, and I^S for $\{x \in I^n : x_i = 0 \text{ for } i \notin S\}$. The complement of S in $\{1, \ldots, n\}$ is written $\widehat{S}$. We often suppress brackets: thus for example $I^{\widehat{i}}$ denotes $\{x \in I^n : x_i = 0\}$. Given a set $A \subset [k]^n$, for $S \subset \{1, \ldots, n\}$ and $x \in [k]^{\widehat{S}}$ we define the *S-section* of A at x to be

$$A_S(x) = \left\{ y \in [k]^S : x + y \in A \right\}.$$

Similarly, the *S-section* of a set $A \subset I^n$ at a point $x \in I^{\widehat{S}}$ is

$$A_S(x) = \left\{ y \in I^S : \ x + y \in A \right\}.$$

1. Minimising the edge-boundary

We shall begin with the problem of minimising the edge-boundary. Our aim is to show that, roughly speaking, the subsets of $[k]^n$ of minimal edge-boundary are of the form $[a]^r \times [k]^{n-r}$ or $[k]^n - \left([a]^r \times [k]^{n-r}\right)$, $r = 1, \dots, n$. More precisely, note that if A is of the form $[a]^r \times [k]^{n-r}$ then $|\partial_e(A)| = |A|^{1-1/r} r k^{(n/r)-1}$. We shall prove that if A is a subset of $[k]^n$ with $|A| \le k^n/2$ then for some $r = 1, \dots, n$ we have $|\partial_e(A)| \ge |A|^{1-1/r} r k^{(n/r)-1}$. Since a set and its complement have the same edge-boundary, it will follow that if $|A| \ge k^n/2$ then for some integer r, $1 \le r \le n$, we have $|\partial_e(A)| \ge (k^n - |A|)^{1-1/r} r k^{(n/r)-1}$.

Our method is to consider a related problem for subsets of the continuous cube I^n; this problem is in fact natural and interesting in its own right. A *brick* in $\mathbf{R}^n$ is a set of the form $\prod_{i=1}^n [a_i, b_i]$, where $a_i < b_i$ for all i. A *rectilinear body* is a finite union of bricks. For a rectilinear body $A \subset I^n$, we write $\sigma(A)$ for the surface area of A in the interior of I^n. Thus

$$\sigma(A) = \sum_{i=1}^n \int_{\widehat{I^i}} |(\partial A)_i(x)| \, dx,$$

where ∂A denotes the usual (topological) boundary of A as a subset of I^n: in $|(\partial A)_i(x)|$ we count the number of boundary points of A inside I^n that lie above x in the ith direction.

We wish to consider the continuous analogue of the problem of minimising the edge-boundary. Given a rectilinear body $A \subset I^n$, at least how large is $\sigma(A)$ in terms of $m(A)$? This can be viewed as the isoperimetric problem for rectilinear subsets of I^n. Our aim is to show that, for a fixed value of $m(A)$, the minimum value of $\sigma(A)$ is attained for a set of the form either $[0, a]^r \times I^{n-r}$ or $I^n - \left((a, 1]^r \times I^{n-r}\right)$, where $1 \le r \le n$.

Let $A \subset I^n$ be a rectilinear body, and $1 \le i \le n$. We define $C_i(A) \subset I^n$, the *i-compression* of A, by giving its i-sections:

$$C_i(A)_i(x) = \begin{cases} \emptyset & \text{if } A_i(x) = \emptyset \\ [0, m(A_i(x))] & \text{otherwise,} \end{cases} \qquad x \in \widehat{I^i}.$$

Then $C_i(A)$ is a rectilinear body, and trivially (or by Fubini's theorem) we have $m(C_i(A)) = m(A)$. We say that A is *i-compressed* if $C_i(A) = A$. Thus A is a down-set iff it is i-compressed for all i.

We have the following easy lemma.

Lemma 1. *(i) Let $A \subset I^n$ be a rectilinear body, and let $1 \le i \le n$. Then $\sigma(C_i(A)) \le \sigma(A)$.*

(ii) Let $A \subset I^n$ be a rectilinear body. Then there is a rectilinear down-set $A' \subset I^n$ such that $m(A') = m(A)$ and $\sigma(A') \le \sigma(A)$.

Proof. (i) For the sake of convenience, write B for $C_i(A)$. For $x \in \widehat{I^i}$ we have $|(\partial B)_i(x)| = 0, 1$ or ∞. If $|(\partial B)_i(x)| = 0$ or 1 then it is easy to see that $|(\partial B)_i(x)| \le |(\partial A)_i(x)|$.

If $|(\partial B)_i(x)| = \infty$ then $m((\partial B)_i(x)) > 0$: say $m((\partial B)_i(x)) = \delta$. It follows that

$$m(B_i(x)) \ge \liminf_{y \to x} m(B_i(y)) + \delta,$$

and so

$$m(A_i(x)) \ge \liminf_{y \to x} m(A_i(y)) + \delta.$$

Thus $m((\partial A)_i(x)) \ge \delta$, so that in particular $|(\partial A)_i(x)| = \infty$.

We have shown that for any $x \in \widehat{I^i}$ we have $|(\partial B)_i(x)| \le |(\partial A)_i(x)|$. By Fubini's theorem it follows that

$$\int_{\widehat{I^i}} |(\partial B)_i(x)|\, dx \;\le\; \int_{\widehat{I^i}} |(\partial A)_i(x)|\, dx.$$

Now fix $j \ne i$. From the above, for $x \in \widehat{I^i}$ we have $m((\partial B)_i(x)) \le m((\partial A)_i(x))$. It follows by Fubini's theorem that for any $y \in \widehat{I^{ij}}$ we have

$$\int_0^1 \Big|(\partial B)_j(y + te_i)\Big|\, dt \;\le\; \int_0^1 \Big|(\partial A)_j(y + te_i)\Big|\, dt,$$

where, as before, e_i denotes the vector with i-coordinate 1 and all other coordinates 0. This implies, again by Fubini's theorem, that

$$\int_{\widehat{I^j}} \Big|(\partial B)_j(x)\Big|\, dx \;\le\; \int_{\widehat{I^j}} \Big|(\partial A)_j(x)\Big|\, dx.$$

Thus $\sigma(B) \le \sigma(A)$.

(ii) If A is i-compressed then so is $C_j(A)$ for any j. It follows that the set $A' = C_n(C_{n-1}(\ldots C_1(A)\ldots))$ is i-compressed for all i, and is therefore a down-set. Certainly $m(A') = m(A)$, and from part (i) we have $\sigma(A') \le \sigma(A)$. ∎

Call a rectilinear body $A \subset I^n$ *extremal* if

$$\sigma(A) = \inf\{\sigma(B) :\ B \subset I^n \text{ a rectilinear body, with } m(B) = m(A)\}.$$

We now introduce some functions associated with the bodies we wish to show are extremal. For $1 \le r \le n$, define functions $f_r, g_r : [0,1] \to \mathbb{R}$ by

$$f_r(v) = rv^{1-1/r}, \quad g_r(v) = r(1-v)^{1-1/r}, \qquad v \in [0,1].$$

Thus if $A = [0,a]^r \times I^{n-r}$ then $\sigma(A) = f_r(m(A))$, and if $A = I^n - \big((a,1]^r \times I^{n-r}\big)$ then $\sigma(A) = g_r(m(A))$.

Define $F_n : [0,1] \to \mathbb{R}$ by

$$F_n(v) = \min\{f_r(v), g_r(v) :\ 1 \le r \le n\}, \quad v \in [0,1].$$

Then $F_n(v) = F_n(1-v)$ for all v. It is easy to check where the minimum is attained: if $2 \le r \le n-1$ then

$$F_n(v) = f_r(v) \qquad \text{for } (1-1/(r+1))^{r(r+1)} \le v \le (1-1/r)^{r(r-1)},$$

while

$$F_n(v) = f_n(v) \qquad \text{for } 0 \le v \le (1-1/n)^{n(n-1)}$$

and

$$F_n(v) = f_1(v) \qquad \text{for } 1/4 \le v \le 3/4.$$

We are now ready to prove the isoperimetric inequality for rectilinear bodies in I^n.

Theorem 2. *Let $A \subset I^n$ be a rectilinear body. Then $\sigma(A) \ge F_n(m(A))$.*

Proof. We proceed by induction on n. The result is trivial for $n = 1$, so we pass to the induction step. By Lemma 1, we may and shall assume that A is a down-set. Furthermore, we may clearly assume that $0 < m(A) < 1$. We shall use some elementary geometric properties of F_n to compare $\sigma(A)$ with $F_n(m(A))$.

Fix $1 \le i \le n$, and for the sake of convenience write $A(z)$ for $A_{\hat{i}}(z)$. Since A is a down-set, we have

$$\int_{I^{\hat{i}}} |(\partial A)_i(x)|\, dx = m(A(0)) - m(A(1)).$$

It follows that

$$\sigma(A) = m(A(0)) - m(A(1)) + \int_0^1 \sigma(A(z))\, dz,$$

and hence by the induction hypothesis we have

$$\sigma(A) \ge m(A(0)) - m(A(1)) + \int_0^1 F_{n-1}(m(A(z)))\, dz.$$

Now, F_{n-1} is a concave function, being the pointwise minimum of a family of concave functions. Moreover, as A is a down-set, $m(A(0)) \ge m(A(z)) \ge m(A(1))$ for all $z \in [0,1]$. Thus, taking $0 < \lambda < 1$ such that

$$m(A) = \lambda m(A(0)) + (1-\lambda) m(A(1)),$$

we have

$$\begin{aligned}\sigma(A) &\ge m(A(0)) - m(A(1)) + \lambda F_{n-1}(m(A(0))) + (1-\lambda)F_{n-1}(m(A(1)))\\ &= \frac{m(A) - m(A(1))}{\lambda} + \lambda F_{n-1}\left(m(A(1)) + \frac{m(A) - m(A(1))}{\lambda}\right)\\ &\qquad + (1-\lambda)F_{n-1}(m(A(1)).\end{aligned}$$

Define $H : [0,1] \to \mathbb{R}$ by

$$H(x) = \frac{m(A) - x}{\lambda} + \lambda F_{n-1}\left(x + \frac{m(A) - x}{\lambda}\right) + (1-\lambda)F_{n-1}(x), \quad x \in [0,1].$$

Then H is the sum of three concave functions, so is itself concave. It follows that

$$\sigma(A) \ge H(m(A(1))) \ge \begin{cases} \min(H(0), H(m(A))) & \text{if } m(A) \ge \lambda \\ \min\left(H\left(\frac{m(A)-\lambda}{1-\lambda}\right), H(m(A))\right) & \text{otherwise.}\end{cases} \tag{1}$$

Let us denote by $E^{(n,v)}$ the set E of the form $[0,a]^r \times I^{n-r}$ or $I^n - \left((a,1]^r \times I^{n-r}\right)$ which satisfies $m(E) = v$, $\sigma(E) = F_n(v)$: if there are two such sets, we select the one with minimal r. For $0 \leq v \leq 1$, define a rectilinear body $C^{(v)}$ by setting

$$C^{(v)} = \begin{cases} E^{(n-1,v)} \times \left[0, \frac{m(A)}{v}\right] & \text{if } v \geq m(A) \\ \left(I^{n-1} \times \left[0, \frac{m(A)-v}{1-v}\right]\right) \bigcup \left(E^{(n-1,v)} \times I\right) & \text{otherwise.} \end{cases}$$

Then inequality (1) states precisely that

$$\sigma(A) \geq \inf\left\{\sigma(C^{(v)}) :\ 0 \leq v \leq 1\right\}.$$

Since $\sigma(C^{(v)})$ is a continuous function of v, the compactness of $[0,1]$ implies that some $C = C^{(v)}$ is extremal.

To complete the proof, we now need only show that C is of the form $[0,a]^r \times I^{n-r}$ or $I^n - \left((a,1]^r \times I^{n-r}\right)$. Since the (closure of the) complement of an extremal set is itself extremal, it is sufficient to consider only the case $C = E^{(n-1,v)} \times [0, m(A)/v]$. In other words, we must show that if the set $C = E^{(n-1,v)} \times [0,a]$ is extremal, with $E^{(n-1,v)} \neq I^{n-1}$ and $0 < a < 1$, then $E^{(n-1,v)} = [0,a]^r \times I^{n-1-r}$ for some r.

From the first part of the proof, we see that $\hat{\imath}$-sections of an extremal set are themselves extremal. It follows that all 2-dimensional sections of C are extremal. Thus if $E^{(n-1,v)} = [0,b]^r \times I^{n-1-r}$, some $r \geq 1$, then $[0,b] \times [0,a]$ is extremal. However, since $a, b < 1$, it is clear that if $b \neq a$ then $\sigma\left([0,a] \times [0,b]\right) > \sigma\left([0,(ab)^{1/2}]^2\right)$, contradicting the extremality of $[0,a] \times [0,b]$, and so we have $b = a$ as required. Finally, if $E^{(n-1,v)} = I^{n-1} - \left((b,1]^r \times I^{n-1-r}\right)$, some $r \geq 2$, then $[0,b] \times [0,a]$ is extremal and also $b > \frac{1}{2}$. However, in that case we have $b = a$, so that $\sigma\left([0,a] \times [0,b]\right) > 1 = \sigma\left(I \times [0,ab]\right)$. This contradiction completes the proof. ∎

By the definition of the function F_n, the inequality of Theorem 2 is best possible. Moreover, for each $m(A)$, there is an extremal set of the form $[0,a]^r \times I^{n-r}$ or $I^n - \left((a,1]^r \times I^{n-r}\right)$, some $1 \leq r \leq n$.

We wish to point out how fortunate it was that, in the above proof, the sets $C^{(v)}$ were down-sets. Indeed, for a down-set A, let us define a new set A' by giving its $\hat{\imath}$-sections:

$$A'(z) = E^{(n-1,m(A(z)))}, \qquad z \in [0,1].$$

Then we *cannot* conclude that $\sigma(A') \leq \sigma(A)$, because A' may not be a down-set, as the sets $E^{(n-1,v)}$, $v \in [0,1]$ are not nested.

We can use the continuous isoperimetric inequality of Theorem 2 to obtain immediately a discrete edge-isoperimetric inequality.

Theorem 3. *Let A be a subset of $[k]^n$ with $|A| \leq k^n/2$. Then*

$$|\partial_e(A)| \geq \min\left\{|A|^{1-1/r} r k^{(n/r)-1} :\ r = 1, \ldots, n\right\}.$$

Proof. Let $B = \bigcup_{x\in A}\prod_{i=1}^{n}[x_i/k,(x_i+1)/k]$. Then $B \subset I^n$ is a rectilinear body, with $m(B) = |A|/k^n$. Moreover, $|\partial_e(A)| = k^{n-1}\sigma(B)$. Hence $|\partial_e(A)| \geq k^{n-1}F_n(|A|/k^n)$, as required. ∎

In particular, we have the following more transparent lower bound on $|\partial_e(A)|$.

Corollary 4. *Let $A \subset [k]^n$. Then*

$$|\partial_e(A)| \geq \begin{cases} 4|A|/k & \text{if } |A| < k^n/4 \\ k^{n-1} & \text{if } k^n/4 \leq |A| \leq 3k^n/4 \\ 4(k^n - |A|)/k & \text{if } |A| > 3k^n/4. \end{cases}$$ ∎

We remark in passing that Theorem 2 can be extended to more general subsets of I^n. Indeed, for any set $A \subset I^n$, let us define the *rectilinear surface area* $\overline{\sigma}(A)$ of A to be

$$\overline{\sigma}(A) = \sum_{i=1}^{n}\int_{\widehat{I^i}}|(\partial A)_i(x)|\,dx.$$

An easy variant of the proof of Theorem 2 gives the following result.

Theorem 5. *Let $A \subset I^n$ be measurable. Then $\overline{\sigma}(A) \geq F_n(m(A))$.* ∎

Although Theorem 3 solves the isoperimetric problem for *rectilinear* subsets of I^n, it would be interesting to know which sets have minimum surface area among *all* subsets of I^n of given volume. More precisely, for a measurable set $A \subset I^n$, write $\sigma(A)$ for the surface area of A in the interior of I^n — defined for example as

$$\sigma(A) = \limsup_{\epsilon\to 0}\,(m(A_\epsilon) - m(A))/\epsilon.$$

Here A_ϵ denotes the *ϵ-neighbourhood* of A : the set of points of I^n which are within Euclidean distance ϵ of a point of A. We conjecture that Euclidean 'cylinders' (or their complements) of some dimension are always best.

Conjecture 6. *Let A be a subset of I^n with $m(A) \leq 1/2$. Then there exist $r \in \{1,\ldots,n\}$ and $a \in [0,n]$ such that the set $B = \left\{x \in I^n : \sum_{i=1}^{r} x_i^2 \leq a\right\}$ satisfies $m(B) = m(A)$ and $\sigma(B) \leq \sigma(A)$.* ∎

To conclude this section, let us see how the analogue of Theorem 3 for the discrete torus $\mathbb{Z}_k^n$ follows from Theorem 2 with almost no additional work. For a rectilinear body $A \subset \mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n$, write $m(A)$ for its volume and $\sigma(A)$ for its surface area.

Theorem 7. *Let $A \subset \mathbb{T}^n$ be rectilinear. Then $\sigma(A) \geq 2F_n(m(A))$.*

Proof. We shall identify A with a rectilinear body in I^n; although this identification is trivial, it has to be done with a little care. Let $\theta : \mathbb{T}^n \to [0,1)^n$ be the bijection induced by the natural bijection from $\mathbb{T}$ to $[0,1)$, and let $\tilde{A} \subset I^n$ be the closure of $\theta(A)$ in I^n. Thus $m\left(\tilde{A}\right) = m(A)$.

For a rectilinear $B \subset I^n$, define

$$\tilde{\sigma}(B) = \sigma(B) + \sum_{i=1}^{n}\left|B_{\hat{i}}(0)\Delta B_{\hat{i}}(1)\right|.$$

Then $\sigma(A) = \tilde{\sigma}\left(\tilde{A}\right)$. Thus, to complete the proof, we must show that for any rectilinear $B \subset I^n$ we have $\tilde{\sigma}(B) \geq 2F_n(m(B))$.

It is easy to check, just as in Lemma 1, that an i-compression does not increase $\tilde{\sigma}(B)$, in other words that $\tilde{\sigma}(C_i(B)) \leq \tilde{\sigma}(B)$ for any $1 \leq i \leq n$. So we may assume without loss of generality that B is a down-set. However, in that case it is clear that $\tilde{\sigma}(B) = 2\sigma(B)$. ∎

The inequality of Theorem 7 is best possible. Indeed, for each $m(A)$, there is an extremal set of the form $[0,a]^r \times \mathbb{T}^{n-r}$ or $\mathbb{T}^n - \left((a,1]^r \times \mathbb{T}^{n-r}\right)$, some $1 \leq r \leq n$.

Just as before, the discrete version of Theorem 7 follows immediately.

Theorem 8. *Let A be a subset of $\mathbb{Z}_k^n$ with $|A| \leq k^n/2$. Then*

$$|\partial_e(A)| \geq \min\left\{2|A|^{1-1/r} r k^{(n/r)-1} : r = 1, \ldots, n\right\}. \quad ∎$$

As we remarked earlier, for the discrete torus the problem of maximising the edge-interior is the same as that of minimising the edge-boundary. Thus a solution to this problem is contained in Theorem 8.

For the isoperimetric problem for *all* subsets of $\mathbb{T}^n$, not just the rectilinear ones, we make the following conjecture, analogous to Conjecture 6. For $u = t + \mathbb{Z} \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, define $|u|$ in the obvious way: $|u| = \min\{|t+s| : s \in \mathbb{Z}\}$.

Conjecture 9. *Let A be a subset of $\mathbb{T}^n$ with $m(A) \leq 1/2$. Then there exist $r \in \{1, \ldots, n\}$ and $a \in [0, n/4]$ such that the set $B = \left\{x \in \mathbb{T}^n : \sum_{i=1}^r |x_i|^2 \leq a\right\}$ satisfies $m(B) = m(A)$ and $\sigma(B) \leq \sigma(A)$.* ∎

2. Maximising the edge-interior — the continuous analogue

We now turn our attention to the problem of maximising the edge-interior. As before, we shall start by considering a related problem for subsets of I^n. For a rectilinear body $A \subset I^n$, and a fixed $k = 2, 3, \ldots$, define $V_k(A)$, the *k-internal volume* of A, to be

$$V_k(A) = \sum_{i=1}^n m\left(\{x \in A : x - te_i \in A \text{ for all } t \in [0, 1/k]\}\right).$$

Thus if A is a down-set then

$$V_k(A) = \sum_{i=1}^n m\left(\{x \in A : x_i \geq 1/k\}\right).$$

The analogue of the problem of minimising the edge-interior is as follows. Among rectilinear bodies in I^n of given volume, which one has maximum k-internal volume? Our aim is to show that, for a fixed value of $m(A)$, the maximum value of $V_k(A)$ is attained for a set of the form either $[0,a]^n$ or $I^n - (a,1]^n$.

Let us start with the following easy lemma, the analogue of Lemma 1.

Lemma 10. *(i) Let $A \subset I^n$ be a rectilinear body, and let $1 \leq i \leq n$. Then $V_k(C_i(A)) \geq V_k(A)$.*

(ii) Let $A \subset I^n$ be a rectilinear body. Then there is a rectilinear down-set $A' \subset I^n$ such that $m(A') = m(A)$ and $V_k(A') \geq V_k(A)$.

Proof. (i) For convenience, write B for $C_i(A)$. It is sufficient to show that for all $1 \leq j \leq n$ and $x \in \widehat{I^i}$ we have

$$\begin{aligned} &m\left(\left\{y \in [0,1] : \ x + ye_i - te_j \in A \text{ for all } t \in [0, 1/k]\right\}\right) \\ &\leq m\left(\left\{y \in [0,1] : \ x + ye_i - te_j \in B \text{ for all } t \in [0, 1/k]\right\}\right). \end{aligned}$$

Fix then an arbitrary $1 \leq j \leq n$ and $x \in \widehat{I^i}$. Suppose first that $j = i$. Then

$$\begin{aligned} &\left\{y \in [0,1] : \ x + ye_i - te_j \in A \text{ for all } t \in [0, 1/k]\right\} \\ &= \{y \in [0,1] : \ y - t \in A_i(x) \text{ for all } t \in [0, 1/k]\}, \end{aligned}$$

and a similar relation holds with B in place of A. Now,

$$m(\{y \in [0,1] : \ y - t \in A_i(x) \text{ for all } t \in [0, 1/k]\}) \leq \max(m(A_i(x)) - 1/k, 0).$$

However, since $B_i(x)$ is an initial segment of $[0, 1]$ we have

$$m(\{y \in [0,1] : \ y - t \in B_i(x) \text{ for all } t \in [0, 1/k]\}) = \max(m(B_i(x)) - 1/k, 0),$$

as required.

Suppose now that $j \neq i$. Then

$$\left\{y \in [0,1] : \ x + ye_i - te_j \in A \text{ for all } t \in [0, 1/k]\right\} = \bigcap_{t \in [0,1/k]} A_i(x - te_j),$$

and a similar relation holds with B in place of A. Now,

$$m\left(\bigcap_{t \in [0,1/k]} A_i(x - te_j)\right) \leq \inf_{t \in [0,1/k]} m(A_i(x - te_j)).$$

However, since the sets $B_i(x - te_j)$, $t \in [0, 1/k]$ are nested we have

$$m\left(\bigcap_{t \in [0,1/k]} B_i(x - te_j)\right) = \inf_{t \in [0,1/k]} m(B_i(x - te_j)),$$

as required.

(ii) The set $A' = C_n(C_{n-1}(\ldots C_1(A)\ldots))$ is a down-set with $m(A') = m(A)$. From part (i) we have $V_k(A') \geq V_k(A)$. ∎

Call a rectilinear body $A \subset I^n$ *k-extremal* if

$$V_k(A) = \sup\{V_k(B) : \ B \subset I^n \text{ a rectilinear body, with } m(B) = m(A)\}.$$

Let us introduce some functions associated with the bodies we wish to show are k-extremal. Define $f, g : [0,1] \to \mathbf{R}$ by

$$f(v) = \begin{cases} nv(1 - 1/kv^{1/n}) & \text{if } v \geq 1/k^n \\ 0 & \text{otherwise,} \end{cases}$$

$$g(v) = \begin{cases} n(1 - 1/k)(1 - (1-v)^{1-1/n}) & \text{if } v \leq 1 - (1 - 1/k)^n \\ n(v - 1/k) & \text{otherwise.} \end{cases}$$

The reason for introducing these functions is that if $A = [0, a]^n$ then $V_k(A) = f(m(A))$, and if $A = I^n - (a, 1]^n$ then $V_k(A) = g(m(A))$.

Put $G_n = \max(f, g)$. Thus $G_n(v) = g(v)$ if $v < 1/k^n$, and $G_n(v) = f(v)$ if $v > 1 - (1 - 1/k)^n$.

We are now ready to show that the bodies mentioned earlier are indeed k-extremal.

Theorem 11. *Let $A \subset I^n$ be a rectilinear body. Then $V_k(A) \geq G_n(m(A))$.*

Proof. We proceed by induction on n. As the result is trivial for $n = 1$, we pass to the induction step. We may clearly assume that $0 < m(A) < 1$, and by Lemma 10 we may assume that A is a down-set. We shall proceed in a manner similar to that in the proof of Theorem 3.

Fix $1 \leq i \leq n$, and for the sake of convenience write $A(z)$ for $A_{\hat{i}}(z)$. Then, by the induction hypothesis, we have

$$V_k(A) = \int_{1/k}^{1} m(A(z))\,dz + \int_{0}^{1} V_k(A(z))\,dz \geq \int_{1/k}^{1} m(A(z))\,dz + \int_{0}^{1} G_{n-1}(m(A(z)))\,dz.$$

Now, G_{n-1} is a convex function, being the pointwise maximum of two convex functions. Also, as A is a down-set, we have $m(A(0)) \geq m(A(z)) \geq m(A(1/k))$ if $0 \leq z \leq 1/k$, and $m(A(1/k)) \geq m(A(z))$ if $z \geq 1/k$. Thus, taking $0 < \lambda < 1/k$ such that

$$\lambda m(A(0)) + ((1/k) - \lambda) m(A(1/k)) = \int_0^{1/k} m(A(z))\,dz$$

and $1/k < \mu < 1$ such that

$$(\mu - 1/k) m(A(1/k)) = \int_{1/k}^{1} m(A(z))\,dz,$$

we have

$$\begin{aligned} V_k(A) &\geq (\mu - 1/k)m(A(1/k)) + \lambda G_{n-1}(m(A(0))) + (\mu - \lambda)G_{n-1}(m(A(1/k))) \\ &= (\mu - 1/k)m(A(1/k)) + \lambda G_{n-1}\left(m(A(1/k)) + \frac{m(A) - \mu m(A(1/k))}{\lambda}\right) \\ &\quad + (\mu - \lambda)G_{n-1}(m(A(1/k))). \end{aligned}$$

Define $H : [0,1] \to \mathbf{R}$ by

$$H(x) = (\mu - 1/k)x + \lambda G_{n-1}\left(x + \frac{m(A) - \mu x}{\lambda}\right) + (\mu - \lambda)G_{n-1}(x)\,, \quad x \in [0,1].$$

Then H is convex, being the sum of three convex functions. It follows that

$$(2)\quad V_k(A) \le H(m(A(1/k))) \le \begin{cases} \max\left(H(0), H\left(\frac{m(A)}{\lambda+\mu}\right)\right) & \text{if } m(A) \le \lambda \\ \max\left(H\left(\frac{m(A)-\lambda}{\mu}\right), H\left(\frac{m(A)}{\lambda+\mu}\right)\right) & \text{otherwise.} \end{cases}$$

Let us denote by $E^{(n,v)}$ the set E of the form $[0,a]^n$, $a \ge 1/k$ or $I^n - (a,1]^n$, $a \le 1/k$ which satisfies $m(E) = v$, $V_k(E) = G_n(v)$, selecting the one of the form $[0,a]^n$ if there are two such sets. For $m(A) \le v \le 1$, define a rectilinear body $C^{(v)}$ by setting

$$C^{(v)} = E^{(n-1,v)} \times [0, m(A)/v],$$

and, for $0 \le h \le \min(1/k, m(A))$ and $\frac{m(A)-h}{1-h} \le v \le \frac{m(A)-h}{(1/k)-h}$, define a rectilinear body $D^{(v,h)}$ by setting

$$D^{(v,h)} = \left(I^{n-1} \times [0,h]\right) \bigcup \left(E^{(n-1,v)} \times (h, h + (m(A) - h)/v]\right).$$

Then inequality (2) states precisely that $V_k(A)$ is at most the supremum of the union of the sets

$$\left\{V_k(C^{(v)}) : \ m(A) \le v \le 1\right\}$$

and

$$\left\{V_k(D^{(v,h)}) : \ 0 \le h \le \min(1/k, m(A)), \ \frac{m(A)-h}{1-h} \le v \le \frac{m(A)-h}{(1/k)-h}\right\}.$$

It follows by a simple compactness argument that some set B of the form $C^{(v)}$ or $D^{(v,h)}$ is k-extremal.

Let us fix a k-extremal set B of the form $C^{(v)}$ or $D^{(v,h)}$. To complete the proof, we need only show that B is of the form $[0,a]^n$ or $I^n - (a,1]^n$. From the first part of the proof, we see that $\hat{i}$-sections of a k-extremal set are themselves k-extremal. It follows that all 2-dimensional sections of B are k-extremal.

We distinguish two cases, according to the form of B. Suppose first that $B = C^{(v)}$ for some v. If B is of the form $\left(I^{n-1} - (a,1]^{n-1}\right) \times [0,b]$ then $I \times [0,b]$ and $[0,a] \times [0,b]$ are k-extremal. However, it is easily checked that a set of the form $[0,c] \times [0,d]$ cannot be k-extremal if $c \ne d$, and so we obtain $a = b = 1$, contradicting $m(A) < 1$. Thus B is of the form $[0,a]^{n-1} \times [0,b]$. However, in that case $[0,a] \times [0,b]$ is k-extremal, whence $a = b$ as required.

Suppose now that $B = D^{(v,h)}$ for some v and h. If B is of the form $(I^{n-1} \times [0,b]) \bigcup \left([0,a]^{n-1} \times (b,c]\right)$ then either $a < 1$ and $I \times [0,b]$ is k-extremal, or $a = 1$ and $I \times [0,c]$ is k-extremal: each of these contradicts $m(A) < 1$. Thus B is of the form $(I^{n-1} \times [0,b]) \bigcup \left(\left(I^{n-1} - (a,1]^{n-1}\right) \times (b,c]\right)$. But then $I \times [0,c]$ is k-extremal, so that $c = 1$. Also, $I^2 - ([0,b] \times [0,a])$ is k-extremal, and so $a = b$, as required. ∎

By the definition of G_n, the inequality of Theorem 11 is best possible. Indeed, for each $m(A)$, there is an extremal set of the form $[0,a]^n$ or $I^n - (a,1]^n$.

We wish to point out that, just as in the proof of Theorem 2, it was very fortunate that the sets $C^{(v)}$ and $D^{(v,h)}$ were down-sets.

We remark that Theorem 11 can be extended to more general subsets of I^n. Indeed, the following result can be proved in essentially the same way as Theorem 11.

Theorem 12. *Let $A \subset I^n$ be measurable. Then $V_k(A) \geq G_n(m(A))$.* ∎

As with Theorem 2, the discrete form of Theorem 11 follows immediately.

Theorem 13. *Let A be a subset of $[k]^n$. Then*

$$|\mathrm{Int}_e(A)| \leq \max\left(n|A|\left(1 - |A|^{-1/n}\right), \; nk^n(1-1/k)\left(1 - (1 - |A|/k^n)^{1-1/n}\right)\right).$$

Proof. Let $B = \bigcup_{x \in A} \prod_{i=1}^n [x_i/k, (x_i+1)/k]$. Then $B \subset I^n$ is a rectilinear body, with $m(B) = |A|/k^n$. Moreover, $|\mathrm{Int}_e(A)| = k^n V_k(B)$. Hence $|\mathrm{Int}_e(A)| \leq k^n G_n(|A|/k^n)$, as required. ∎

Note in particular that if $|A| \geq k^n\left(1-(1-1/k)^n\right)$ in Theorem 13 then $|\mathrm{Int}_e(A)| \leq n|A|\left(1 - |A|^{-1/n}\right)$.

Unfortunately, although the continuous inequality of Theorem 11 is best possible, the discrete inequality of Theorem 13 is not close to best possible for every value of $|A|$. The reason for this is that the extremal sets for the continuous problem which are not of the form $[0,a]^n$ are all of the form $I^n - (b,1]^n$, where $b < 1/k$. They therefore do not correspond to any discrete sets, as they cannot be approximated by unions of bricks of the form $\prod_{i=1}^n [x_i/k, (x_i+1)/k]$. Indeed, for this reason we cannot even conclude from Theorem 11 that the extremal sets for the discrete problem of maximising the edge-interior are not nested. This is in contrast to the situation with the extremal sets for the continuous analogue of the problem of minimising the edge-boundary, as given by Theorem 2.

In fact, in the next section we shall use different methods to prove an *exact* discrete inequality. The extremal sets for that inequality will turn out to be nested.

3. Maximising the edge-interior

Our aim in this section is to give a best possible solution to the problem of maximising the edge-interior in the grid, thereby improving upon Theorem 13. Our method will be purely discrete.

Define an order on $[k]^n$, the *cube order* , by letting $x = (x_i)_1^n$ precede $y = (y_i)_1^n$ if for some $s \in [k]$ we have $\{i : x_i = s\} < \{i : y_i = s\}$ in the binary order on $\mathcal{Q}^n$, with $\{i : x_i = t\} = \{i : y_i = t\}$ for all $t > s$. Equivalently, x precedes y if and only if $w(x) < w(y)$, where $w(x) = \sum_{i=1}^n 2^{i+nx_i}$. Thus for example for each $s \leq k$ the set $[s]^n = \left\{x \in [k]^n : x_i < s \text{ for all } i\right\}$ is an initial segment of the cube order on $[k]^n$.

Our aim in this section is to show that initial segments of the cube order on $[k]^n$ have maximum edge-interior among subsets of $[k]^n$ of given cardinality. Our main tool will be the notion of an i-symmetrisation, which we now describe.

For $S \subset \{1, \ldots, n\}$, the *cube order* on $[k]^S$ is just the restriction to $[k]^S$ of the cube order on $[k]^n$. It is easy to see that if A is an initial segment of the cube order on $[k]^n$ then each $\widehat{i}$-section of A is an initial segment of the cube order on $[k]^{\widehat{i}}$.

For any set $A \subset [k]^n$, and $1 \leq i \leq n$, we define a set $S_i(A) \subset [k]^n$, the *i-symmetrisation* of A, by giving its $\widehat{i}$-sections:

$$S_i(A)_{\widehat{i}}(x) = C^{(i)}(|A_{\widehat{i}}(x)|), \qquad x \in [k],$$

where $C^{(i)}(m)$ denotes the set of the first m elements in the cube order on $[k]^{\widehat{i}}$. In other words, S_i replaces each $\widehat{i}$-section of A by an initial segment of the cube order on $[k]^{\widehat{i}}$ of the same size. We say that A is *i-symmetrised* if $S_i(A) = A$. Thus for example every initial segment of the cube order on $[k]^n$ is i-symmetrised for all i.

What can we say about a set $A \subset [k]^n$ which is i-symmetrised for all i? Such a set is certainly a down-set (for $n \geq 2$), and if $n = 2$ then that is all we can say. For $n \geq 3$, however, being i-symmetrised for every i is a very restrictive condition. One might even expect that, for $n \geq 3$, every set $A \subset [k]^n$ which is i-symmetrised for all i is an initial segment of the cube order. Unfortunately, a moment's thought shows that this is not the case — a simple counter-example is $\left\{x \in [k]^3 : \sum x_i \leq 1\right\}$.

However, we do have the following simple lemma, which shows that, for $n \geq 3$, a subset of $[k]^n$ which is i-symmetrised for all i is not too far from being an initial segment of the cube order on $[k]^n$; in fact, it differs from an initial segment along at most two lines. To make the lemma more palatable, note that if $A \subset [k]^n$ is i-symmetrised for every i, and for some r and s we have

$$\{x \in [s+1]^n : x_i \leq s-1 \text{ for } i \geq r\} \subset A \subset \{x \in [s+1]^n : x_i \leq s-1 \text{ for } i \geq r+1\},$$

then A is an initial segment of the cube order on $[k]^n$.

Lemma 14. *Let $n \geq 3$, and let A be a non-empty subset of $[k]^n$ which is i-symmetrised for all i. Let s be the minimal integer such that $A \subset [s+1]^n$, and set*

$$r = \max\{1 \leq i \leq n : x_i = s, \text{ some } x \in A\}.$$

(i) If $r = 1$ then

$$A \supset [s]^n - \Big(\{(z, s-1, s-1, \dots, s-1) : z \geq 0\} \bigcup \{(s-1, z, s-1, s-1, \dots, s-1) : z \geq 1\}\Big).$$

(ii) If $1 < r < n$ then

$$A \supset \{x \in [s+1]^n : x_i \leq s-1 \text{ for } i \geq r\} - \{(s, s, \dots, s, s-1, z, s-1, s-1, \dots, s-1) : z \geq 1\},$$

where in the last expression the z is in position $r+1$.

(iii) If $r = n$ then

$$A \supset \{x \in [s+1]^n : x_n \leq s-1\} - \Big(\{(s, s, \dots, s, z) : z \geq 0\} \bigcup \{(z, s, s, \dots, s, s-1) : z \geq 1\}\Big).$$

Proof. (i) Let $y = \sum_{i=1}^n (s-1)e_i$. We have $se_1 \in A$, and so the fact that A is 2-symmetrised gives $y - (s-1)e_2 \in A$. Thus, as A is a down-set, it is sufficient to show that $y - e_i \in A$ for all $i \geq 3$ and that $y - e_1 - e_2 \in A$.

Since $y-(s-1)e_2 \in A$, and A is 3-symmetrised, we have $y-e_1-e_2 \in A$. Also, for each $i \geq 3$, the fact that A is 1-symmetrised gives $y-e_i \in A$.

(ii) Let $y=\sum_{i=1}^{r-1} se_i + \sum_{i=r}^{n}(s-1)e_i$. Since $se_r \in A$, and A is n-symmetrised, we have $y-(s-1)e_n \in A$. For each $i<r$, the fact that A is r-symmetrised yields $y-e_i \in A$. So it remains to show that $y-(s-1)e_{r+1} \in A$ and that $y-e_i \in A$ for all $i \geq r$, $i \neq r+1$.

As A is $(r+1)$-symmetrised, $se_r \in A$ implies $y-(s-1)e_{r+1} \in A$. Since A is r-symmetrised, it follows that $y-e_i \in A$ for each $i \geq r+2$. Also, as A is 1-symmetrised, we have $y-e_r \in A$.

(iii) Let $y=\sum_{i=1}^{n-1} se_i + (s-1)e_n$. Since $se_n \in A$ and A is 1-symmetrised, we have $y-se_1 \in A$. So it remains to show that $y-e_i \in A$ for all $2 \leq i \leq n-1$. However, this follows from $y-se_1 \in A$ and the fact that A is n-symmetrised. ∎

Armed with Lemma 14, we are now ready to give a best possible upper bound for the size of the edge-interior of a subset of $[k]^n$ of given cardinality.

Theorem 15. *Let $A \subset [k]^n$, and let J be the set of the first $|A|$ elements in the cube order on $[k]^n$. Then $|\mathrm{Int}_e(A)| \leq |\mathrm{Int}_e(J)|$.*

Proof. We proceed by induction on n. The result is trivial for $n=1$, but we shall start by proving the result for $n=2$.

For $A \subset [k]^2$, put $p_1 = |\{x_1 : x \in A\}|$ and $p_2 = |\{x_2 : x \in A\}|$. Then the number of edges in direction e_i spanned by A is at most $|A|-p_i$, and so $|\mathrm{Int}_e(A)| \leq 2|A|-(p_1+p_2)$. Now, $|A| \leq p_1p_2$, and so for any $r=0,\ldots,k-1$ it follows by the arithmetic-geometric mean inequality that $|A|>r^2$ implies $p_1+p_2>2r$, while $|A|>r(r+1)$ implies $p_1+p_2>2r+1$. Hence $|\mathrm{Int}_e(A)| \leq |\mathrm{Int}_e(J)|$, as required.

We now turn our attention to the induction step. We first wish to show that for any $A \subset [k]^n$ and $1 \leq i \leq n$ we have $|\mathrm{Int}_e(S_i(A))| \geq |\mathrm{Int}_e(A)|$: in other words, that an i-symmetrisation does not decrease the edge-interior.

For convenience, write B for $S_i(A)$. Let us write $A(z)$ for $A_{\hat{i}}(z)$ and $B(z)$ for $B_{\hat{i}}(z)$. We have

$$|\mathrm{Int}_e(A)| = \sum_{z=0}^{k-1} |\mathrm{Int}_e(A(z))| + \sum_{z=0}^{k-2} |A(z) \cap A(z+1)|$$

and

$$|\mathrm{Int}_e(B)| = \sum_{z=0}^{k-1} |\mathrm{Int}_e(B(z))| + \sum_{z=0}^{k-2} |B(z) \cap B(z+1)|.$$

Now, by the induction hypothesis we have $|\mathrm{Int}_e(B(z))| \geq |\mathrm{Int}_e(A(z))|$ for all z. Moreover, for each z the sets $B(z)$ and $B(z+1)$ are nested, and so $|B(z) \cap B(z+1)| \geq |A(z) \cap A(z+1)|$. It follows that $|\mathrm{Int}_e(B)| \geq |\mathrm{Int}_e(A)|$, as required.

Given $A \subset [k]^n$, let $G = \{B \subset [k]^n : |B|=|A| \text{ and } |\mathrm{Int}_e(B)| \geq |\mathrm{Int}_e(A)|\}$. To complete the induction step, we need to show that G contains an initial segment of the cube order on $[k]^n$. Choose $B \in G$ with $W(B)=\sum_{x\in B} 2^{-w(x)}$ maximal. We claim that this B is an initial segment of the cube order on $[k]^n$.

For any $1 \leq i \leq n$, we have $|S_i(B)|=|B|$, and $|\mathrm{Int}_e(S_i(B))| \geq |\mathrm{Int}_e(B)|$. Hence $S_i(B) \in G$. Moreover, $W(S_i(B)) \geq W(B)$, with equality if and only if $S_i(B)=B$. It follows by the definition of B that B is i-symmetrised for all i. Let s be the minimal integer such that $B \subset [s+1]^n$, and set

$$r = \max\{1 \leq i \leq n : \; x_i = s, \; \text{some } x \in B\}.$$

We distinguish three cases, according to the value of r.

(i) Suppose first that $r = 1$. To show that B is an initial segment of the cube order on $[k]^n$, it is sufficient to show that $[s]^n \subset B$. By Lemma 14, we know that B contains

$$[s]^n - \Big(\{(z, s-1, s-1, \ldots, s-1) : z \geq 0\} \bigcup \{(s-1, z, s-1, s-1, \ldots, s-1) : z \geq 1\}\Big).$$

Let $y = \sum_{i \geq 3}(s-1)e_i$. We claim first that $y + (s-1)e_1 + (s-2)e_2 \in B$. Indeed, if this is not the case then put

$$a = \max\{z \in [k] : \; y + (s-1)e_1 + ze_2 \in B\},$$

so that $0 \leq a \leq s-3$. Among those $x \in B$ with $x_1 = s$, choose one with $\sum x_i$ maximal, and set

$$B' = B \cup \{y + (s-1)e_1 + (a+1)e_2\} - \{x\}.$$

Then $|B'| = |B|$, and by the choices of a and x we have $|\mathrm{Int}_e(B')| \geq |\mathrm{Int}_e(B)|$. However, since $W(B') > W(B)$, this contradicts the definition of B. Thus $y + (s-1)e_1 + (s-2)e_2 \in B$.

We now claim that $y + (s-1)e_1 + (s-1)e_2 \in B$. Indeed, suppose not. If $y + (s-1)e_2 \in B$ then, exactly as above, we would obtain a set $B' \in G$ with $W(B') > W(B)$, and so $y + (s-1)e_2 \notin B$. Among those $x \in B$ with $x_1 = s$ and $x_n = 0$, choose one with $\sum x_i$ maximal, and put

$$p = |\{z \in [k] : \; x + ze_n \in B\}|,$$

so that $1 \leq p \leq s$. Set

$$B' = B \cup \{y + (s-1)e_2 + ze_1 : \; 0 \leq z < p\} - \{x + ze_n : \; z \in [k]\}.$$

Then $|B'| = |B|$, and it is easy to see that $|\mathrm{Int}_e(B')| \geq |\mathrm{Int}_e(B)|$. But $W(B') > W(B)$, contradicting the definition of B.

Hence $(s-1, s-1, \ldots, s-1) \in B$, and so $[s]^n \subset B$.

(ii) Suppose now that $1 < r < n$. To show that B is an initial segment of the cube order, it suffices to show that $\{x \in [s+1]^n : \; x_i \leq s-1 \text{ for } i \geq r\} \subset B$. We shall proceed in a manner similar to case (i) above. By Lemma 14 we know that B contains

$$\{x \in [s+1]^n : \; x_i \leq s-1 \text{ for } i \geq r\} - \{y + ze_i : \; z \geq 1\},$$

where $y = \sum_{i<r} se_i + (s-1)e_r + \sum_{i \geq r+2}(s-1)e_i$.

We claim that $y + (s-1)e_{r+1} \in B$. Indeed, if this not the case then put

$$a = \max\{z \in [k] : \; y + ze_{r+1} \in B\},$$

so that $0 \leq a < s-1$. Among those $x \in B$ with $x_r = s$, choose one with $\sum x_i$ maximal, and set

$$B' = B \cup \{y + (a+1)e_{r+1}\} - \{x\}.$$

Then $|B'| = |B|$, and $|\mathrm{Int}_e(B')| \geq |\mathrm{Int}_e(B)|$, so that $B' \in G$. But $W(B') > W(B)$, contradicting the definition of B.

(iii) Suppose finally that $r = n$. To show that B is an initial segment of the cube order, it is sufficient to show that $\{x \in [s+1]^n : x_n \leq s-1\} \subset B$. By Lemma 14, we know that B contains

$$\{x \in [s+1]^n : x_n \leq s-1\} - \Big(\{(s,s,\ldots,s,z) : z \geq 0\} \bigcup \{(z,s,s,\ldots,s,s-1) : z \geq 1\}\Big).$$

Exactly as in case (i) above, it follows first that $(s-1,s,s,\ldots,s,s,s-1) \in B$ and then that $(s,s,\ldots,s,s-1) \in B$. Hence $\{x \in [s+1]^n : x_n \leq s-1\} \subset B$, as required.

This completes the induction step, and so completes the proof of Theorem 15. ∎

In particular, we have the following corollary.

Corollary 16. *Let $A \subset [k]^n$ satisfy $|A| \leq s^n$, some $s = 1,\ldots,n$. Then the subgraph of $[k]^n$ induced by A has average degree at most $2n(1-1/s)$.* ∎

Corollary 16 immediately implies the corresponding result about induced subsets of the infinite graph $\mathbb{Z}^n$: a subgraph of $\mathbb{Z}^n$ with at most s^n vertices has average degree at most $2n(1-1/s)$. The sets $[s]^n$ show that this bound is attained for every s.

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